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Reflections on Quantum Logic

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This is a short, self-contained summary of problems connected with the interpretation of state vectors in quantum mechanics. We discuss the reconstruction of the "ψ function" from statistical data, some related mathematical questions, the classical "paradoxes," the probability interpretation of the state vectors, and, finally, quantum logic in relation to hidden variable theories and Hilbert space formalism, to build up a consistent framework for the indeterministic quantum picture of nature.

We collect here remarks concerning the interpretation of state vectors in quantum mechanics. The well-known "paradoxes" are avoided by a systematic use of the Heisenberg picture, so that a state vector is considered as a code for the maximal information about the system available to an external observer, while the internal logic of a quantum system is described by the lattice of closed subspaces of the underlying Hilbert space. We discuss an interesting mathematical problem concerning the reconstruction of a state vector from statistical data alone, and close the paper with a brief description of quantum logic and hidden variable theories that throws further light on the parallelism and distinction between the quantum and classical pictures of nature.

(1) To fix our notation we recall the standard formalism of quantum mechanics. With a quantum system Q one associates a complex Hilbert space X and a set $\mathcal{A}$ of Hermitian operators in X; $(x|y)$ is the inner product of vectors x, y of X. Let $\bar{X}$ be the set of rays, or one-dimensional subspaces, of X. The elements of $\bar{X}$ are called *states* of Q, the elements of $\mathcal{A}$ are called *observables*. Let $\mathcal{P}(X)$ be the lattice of all projectors in X and $\mathcal{B}(R)$ denote the algebra of Borel subsets on the real line R. To every Hermitian operator A there corresponds a projection-valued measured P_A: $\mathcal{B}(R) \rightarrow \mathcal{P}(X)$ in

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$\mathfrak{B}(R)$. The states $\bar{x} \in \bar{X}$ are used as *codes* for the maximal information about the way the system Q has been prepared. To any observable $A \in \mathfrak{A}$ there corresponds a macroscopic instrument $I(A)$ measuring its value when interacting with Q. Suppose that a sequence $s = \{s_1, s_2, s_3, \dots\}$ of *identically* prepared copies of Q interacts with $I(A)$ and assume that $I(A)$ is restored to its initial position after the interaction with s_n is over and before the interaction with s_{n+1} starts. For any $B \in \mathfrak{B}(R)$ the instrument $I(A)$ checks the statement "the value of A belongs to B" denoted, for brevity, by "$v(A) \in B$"; we set $\alpha_i = 0$ (or 1) whenever $I(A)$ finds $v(A) \in B$ false (or true) on s_i. Born's postulate asserts that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_i = (P_A(B)x|x)/(x|x)$$

for any $x \in \bar{x}$, where $\bar{x}$ denotes the state in which the elements of s have been prepared. Generalizing this setup one introduces *mixed states*, or *mixtures*, as positive Hermitian operators Y of trace class with $\operatorname{Sp} Y = 1$. By a well-known lemma, any such operator can be represented in the form $Y = \Sigma_{i=1}^{\infty} \lambda_i P_i$, with $P_i \in \mathfrak{P}(X), \operatorname{Sp} P_i = 1, \Sigma_{i=1}^{\infty} \lambda_i = 1$, $\lambda_i \geq 0$ for any i. In particular, P_i is a projector on an one-dimensional subspace $\bar{x}_i \in \bar{X}$. A mixture Y is used as a code for *noncomplete* information about the way of preparation of Q. The sequence $s = \{s_1, s_2, \dots\}$ of copies of Q is said to be prepared in a mixed state Y, if it can be subdivided into subsequences $s^{(\kappa)} = \{s_1^{(\kappa)}, s_2^{(\kappa)}, \dots\}$ in such a way that $s = \cup_{\kappa=1}^{\infty} s^{(\kappa)}$, $s_j^{(\kappa)}$ is prepared in the state $\bar{x}_\kappa$ and λ_κ denotes the relative density of $s^{(\kappa)}$ in s, that is, $\lambda_k = \lim_{N \to \infty} n^{(\kappa)}/N$, where $n^{(\kappa)}$ denotes the number of $s_i \in s^{(\kappa)}$ with $i \leq N$. In this situation Born's postulate (1) implies that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_i = \operatorname{Sp}[Y \cdot P_A(B)] \tag{2}$$

where $\alpha = \{\alpha_1, \alpha_2, \dots\}$ is the result of checking by $I(A)$ the validity of $v(A) \in B$ on s, as above. An interaction of a quantum system Q_1 with another system Q_2, in particular, with any of the instruments $I(A)$ introduced before, can be described by considering the combined system $Q = \{Q_1, Q_2\}$ with the underlying Hilbert space $X = X_1 \otimes X_2$ and the set of observables containing any of the observables $A_1 \otimes 1$ and $1 \otimes A_2$, whenever $A_i \in \mathfrak{A}_i$, where $X_i, \mathfrak{A}_i$ are the Hilbert space and the set of observables of Q_i. Observables of Q are assumed to be time dependent. Namely, one introduces an Hermitian operator H in X, the Hamiltonian of interaction, and, for any $A \in \mathfrak{A}$ measured by $I(A)$ at the moment of *time* t on a sample s of copies of Q prepared in the state Y, replaces (2) by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_i = \mathrm{Sp}\left[Y \cdot P_{A_t}(B)\right] \tag{3}$$

where $A_t = U_t^{-1} A U_t$, $U_t = \exp(iHt)$ is the evolution operator of the system. It goes without saying that the measurements are supposed to be performed so quickly that A_t could not change essentially during the interval of the measurement.

Sometimes it is convenient to replace Q_2 by an external (to Q_1) potential. In this case we work in X_1 only, so that H is a chosen Hermitian operator in this space.

Moreover, if the expression in the right-hand side of (1) is actually less than 1 and greater than 0, one assumes that the sequence $\alpha = \{\alpha_1, \alpha_2, \ldots\}$ obtained by $I(A)$ is *random*: it should not possess any regularities and is subject to the restriction (1) only. Our first remark is that the work on the right definition of random sequences (see, e.g., Benioff, 1977, and references therein) can be rather relevant in this context. The point is that here the random character of the sequence α is due to the nature of things, rather than to our incomplete knowledge of it (as is assumed to be the case in classical statistical physics). We are not quite sure that the classical theory of probabilities is an adequate tool in this situation; if, however, it is used, one should notice that the Born postulate, as stated above, is a stronger statement than "probability of $v(A) \in B$ is equal to $(P_A(B)x|x)/(x|x)$," since (1) is required to hold for the particular sequence α obtained by $I(A)$. Our further more technical comments follow in Sections (2)–(4) below.

(2) One may ask oneself whether the notion of a state vector is operationally well defined. Even assuming that one can carry out a long enough series of measurements to obtain a good approximation to the value $(P_A(B)x|x)/(x|x)$ [Born's postulate says nothing about the rate of convergence in the left hand side of (1)!], do such data suffice to reproduce $\bar{x}$? To state it rigorously, we call a set $\mathcal{A}$ of Hermitian operators in X sufficient, if

$$\frac{(P_A(B)x_1|x_1)}{(x_1|x_1)} = \frac{(P_A(B)x_2|x_2)}{(x_2|x_2)} \quad \text{for any } A \in \mathcal{A},\ B \in \mathfrak{B}(R) \tag{4}$$

implies $\bar{x}_1 = \bar{x}_2$. We ask now: is the set of observables of Q sufficient? The following simple example shows that the answer may be negative.

Example. Consider a free massive particle without internal degrees of freedom; choose $X = \mathfrak{L}^2(R^3)$, the space of complex-valued square integrable functions on R^3, $\mathcal{Q}=\{\hat{p}_i, \hat{x}_i, \hat{L}_i | 1\leq i\leq 3\}$, where, as usual,

$$(\hat{x}_j f)(\mathbf{x}) = x_j f(\mathbf{x}), \qquad (\hat{p}_j f)(\mathbf{x}) = \left(-i\frac{\partial}{\partial x_j} f\right)(\mathbf{x}), \qquad \hat{L}_j = \hat{x}_i \hat{p}_k - \hat{x}_k \hat{p}_i$$

x_j denotes the jth projection of $\mathbf{x}\in R^3$, (j, i, k) is an even permutation of indices $(1,2,3)$.

Lemma. The set $\mathcal{Q}$ is not sufficient.

Proof. Choose $\psi_1(\mathbf{x}) = f(r)\exp[i\varphi(r)]$, $\psi_2(\mathbf{x}) = f(r)\exp[-i\varphi(r)]$, where $r=(x_1^2+x_2^2+x_3^2)^{1/2}$; one can check easily that, although (4) is satisfied, $\psi_1 \neq \lambda\psi_2$ for any complex constant λ whenever φ is not constant (f and φ are two *real*-valued functions). ■

To repair the situation one may make our system interact with different potentials and ask whether measuring A_t for several t suffices to reproduce $\bar{x}$. This question has been studied, in the situation of the example, in Kreinovich (1977); we summarize here its results.

Definition. A system of potentials $\{V_n | n=1,2,\dots\}$ is called p-sufficient (x-sufficient) on the class K, if $|\exp(-iH_n t)\hat{\psi}_1(p)| = |\exp(-iH_n t)\hat{\psi}_2(p)|$ for all p, n, t [$|\exp(-iH_n t)\psi_1(x)| = |\exp(-iH_n t)\psi_2(x)|$ for all x, n, t] implies that $\bar{\psi}_1 = \bar{\psi}_2$ for any $\psi_1, \psi_2 \in K$. Here $H_n = p^2/2m + V_n(x)$ in the notation of the example, $\hat{\psi}$ denotes the Fourier transform of ψ.

Theorem. (Kreinovich, 1977) (1) If $\{V_n\}$ is L^2-dense, it is p-sufficient in the class of all L^2 functions with continuous Fourier transform; (2) if $\{\mathbf{x}_i\}$ is everywhere dense in R^3 and $\alpha \geq 0$, any of the systems $\{|x-x_i|^{-1}\exp(-\alpha|x-x_i|)\}$, $\{|x-x_i|^{-(2n+1)}\}$ is p-sufficient in the same class of functions; (3) there exists a p-sufficient system with no finite p-sufficient subsystem; (4) there exists a system of eight potentials p-sufficient on the class of L^2-functions with analytic Fourier transform.

One should expect that this theorem can be improved, since to reproduce one complex-valued function ψ it requires data about an infinite sequence of functions generated by it. Unfortunately, too straightforward arguments fail (a typical example of an erroneous argument from a classical textbook (Kemble, 1937, p. 71): write $\psi(x,t) = \rho(x,t)\exp[i\varphi(x,t)]$, it follows then that $\partial^2\rho/\partial t^2 = \text{const}\,\nabla(\rho^2\nabla\varphi)$, suppose now that ρ and $\partial^2\rho/\partial t^2$ are known, then φ is determined as a solution of the equation $\nabla(\rho^2\nabla\varphi) = f$; unfortunately, there are a lot of solutions of this equation, since φ does not satisfy a priori any boundary conditions).

The following hypothesis seems to be true, but its proof (or disproof) is unknown to us.

Hypothesis. The set $\mathcal{Q} = \{A_1, A_2, A_3\}$ of three Hermitian operators in X is sufficient, if no pair A_i, A_j with $i \neq j$ has a common invariant subspace.

We can prove only a much weaker result.

Proposition. For any Hermitian operator A_1 with a simple purely discrete spectrum there exist two other Hermitian operators A_2, A_3 with purely discrete spectra such that the conclusion of the hypothesis holds.

Proof. Let $x \in X, (x|x) = 1, \{l_1, l_2, \dots\}$ is the basis of normalized eigenvectors for A_1 and $x = \sum_{i=1}^{\infty} x_i l_i$; since $|x_i|$ are given, we can rewrite the last equation as $x = \sum_{k=1}^{\infty} \exp(i\varphi_k) f_k$ with real φ_k to be determined. We choose A_2 and A_3 to have simple spectra with eigenvectors of the form

$$\frac{l_1 + l_2}{\sqrt{2}}, \quad \frac{l_1 + l_2}{\sqrt{2}}, \dots \quad \text{and} \quad \frac{l_1 + il_2}{\sqrt{2}}, \quad \frac{l_1 + il_3}{\sqrt{2}}, \dots$$

so that

$$A_2(l_1 + l_k) = \lambda_k^{(2)}(l_1 + l_k), \; A_3(l_1 + il_k) = \lambda_k^{(3)}(l_1 + il_k)$$

Then

$$x = \sum_{k=1}^{\infty} (x|g_k) g_k = \sum_{k=1}^{\infty} (x|h_k) h_k$$

where

$$g_k = (l_1 + l_{k+1})/\sqrt{2}, \qquad h_k = (l_1 + il_{k+1})/\sqrt{2}$$

it follows that

$$(x|g_k) = [\exp(i\varphi_{k+1})(f_{k+1}|l_{k+1}) + \exp(i\varphi_1)(f_1|l_1)]/\sqrt{2}$$
$$= \alpha e^{i\varphi_{k+1}} + \beta e^{i\varphi_1}, \qquad (x|h_k) = \gamma e^{i\varphi_{k+1}} + i\delta e^{i\varphi_1}$$

where $\alpha, \beta, \gamma, \delta$ are given real numbers. Now $|(x|g_k)|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\cos(\varphi_{k+1} - \varphi_1)$, $|(x|h_k)|^2 = \gamma^2 + \delta^2 + 2\gamma\delta\sin(\varphi_{k+1} - \varphi_1)$ determine all the differences $(\varphi_k - \varphi_1) \bmod 2\pi$ [and therefore the vector x up to a factor $\exp(i\varphi)$], as soon as $|(x|g_k)|^2$ and $|(x|h_k)|^2$ are given. ■

(3) We turn now to the question of physical consistency of quantum mechanics. Two points deserve special attention: firstly, would not the assumptions made in Section (1) contradict the experimentally well-established results of classical physics when applied to macroscopic system? Secondly, can the relativistic invariance be restored? These questions are being discussed here in the context of two well-known paradoxes (Einstein et al., 1935; Schrödinger, 1935): Schrödinger's cat paradox (SCP) and the Einstein–Podolsky–Rosen paradox (EPR). In SCP one considers a macroscopic device M with a pointer M_0 interacting with a quantum system Q and designed in such a way that the position of the pointer (after the interaction is over, say, for any moment of time $t>t_0$) displays the value of the observable A of Q measured in the course of the experiment. To simplify the discussion let us assume that the spectrum of A contains only two points, say, ± 1. The pointer may be found in one of the three positions $0, \pm 1$; moreover, let its initial position be 0, while for $t>t_0$ its position, 1 or -1, coincides with the measured value of A. Let P_t be the observable of the compound system $\{Q, M\}$ corresponding to the position of M_0 at the moment t. Let us assume that the interaction starts at the moment t_1 and let $\bar{x}$ describe the way of preparation of M (we assume $\bar{x}$ to be a pure state, for simplicity); let us assume that Q has been prepared in a state $\bar{y}$, where $y=y_1+y_2$ and $Ay_1=y_1, Ay_2=-y_2$. Then $P_t(x\otimes y)=0$ for $t<t_1$ and $P_t(x\otimes y)=x\otimes(y_1-y_2)$ for $t>t_0$. Now a macroscopic observable P_t is supposed to have a definite value for large enough t (the pointer M_0 is to be found in one of its finite positions ± 1!), but we cannot predict it theoretically. Reading off the position of M_0 changes our information about the system $\{Q, M\}$. One may ask whether it is possible, in practice, to distinguish between the preparation procedures described by the pure state $\overline{x\otimes y}$and by the mixture $\frac{1}{2}(R_1+R_2)$, where R_i is the projector on $\overline{x\otimes y_i}$. To settle this question let us consider an observable B_t and compare its average values $\langle B_t\rangle_R$ and $\langle B_t\rangle_{\overline{x\otimes y}}$ in the states $R=\frac{1}{2}(R_1+R_2)$ and $\overline{x\otimes y}$. It follows that

$$\langle B_t\rangle_R=\mathrm{Sp}(R\cdot B_t)=\frac{(B_t(x\otimes y)|x\otimes y)}{(x\otimes y|x\otimes y)}=\langle B_t\rangle_{\overline{x\otimes y}}$$

for any B_t such that $B_0\cdot P_0=P_0\cdot B_0$. If B_t is a macroscopic observable, it should allow for simultaneous with P_t measurements, and, therefore, we can assume $B_tP_t=P_tB_t$ in this case. Thus the preparation procedures described by R and by $\overline{x\otimes y}$ are macroscopically indistinguishable, and the situation described in SCP does not differ too much from the usual nondeterministic situations of statistical physics. However, on the quantum level the states R and $\overline{x\otimes y}$ are different, and, therefore, the system prepared in the state $\overline{x\otimes y}$

exhibits some subtle properties unexpected for a macroscopic pointer. These quantum properties are destroyed by the observation (reading off the position of the pointer). In order to notice these quantum effects one has to abandon the use of M as a measuring device, since any observable one can measure simultaneously with P_t cannot be used to dinstinguish between R and $\overline{x\otimes y}$.

To describe the Gedankenexperiment considered in EPR we introduce a quantum system Q that consists of two parts in two regions of space, σ_1 and σ_2, far removed from each other (in particular, we assume $\sigma_1 \cap \sigma_2 = \varnothing$). Let us define an observable

$$A_t(x) = \begin{cases} 0, & x \in \sigma_1 \cup \sigma_2 \\ A^1, & x \in \sigma_1 \\ A^2, & x \in \sigma_2 \end{cases}$$

where A^1 and A^2 are operators in the underlying Hilbert space. Consider a preparation procedure $\bar{\psi}$ such that $\psi = \psi_1 + \psi_2$ and $A^1\psi_i = A^2\psi_i = (-1)^i\psi_i$, and suppose that a σ_1 observer measures $A_t(x)$ for some t and $x \in \sigma_1$; he can then predict the outcome of the measurement made in σ_2. Can this procedure be used for an instantaneous transmission of information? Suppose $A_t(x)$ has been measured in σ_1 on a sequence $s = (s_1, s_2, \ldots)$ of $\bar{\psi}$-prepared copies of the system; then the results of $A_t(x)$ measurements for $x \in \sigma_2$ are also known. However, since both a measurement of $A_t(x)$ for $x \in \sigma_2$ on s "before" the experiment in σ_1 and a measurement of the same type "after" the experiment in σ_1 yield the same statistics (i.e., the same $\lim_{n\to\infty} 1/n \sum_{i=1}^{n} \alpha_i$, equal to 0 in our case; here α_i is the value of $A_t(x)$ obtained during the measurement on s_i), no transmission of information occurs (in fact, the situation we have just described does not differ too much from a nonquantum example: suppose one sends two identical balls of unknown color in closed boxes to different regions of space, and two observers far removed from each other open the boxes, then any of them is aware of the answer another one gets). A contradiction would arise if one of the observers could check the value of an observable distinguishing between $\bar{\psi}$ and $\frac{1}{2}(P_1 + P_2)$, where P_i is the projector on $\bar{\psi}_i$. Thus we set now

$$A_t(x) = \begin{cases} 0, & x \in \sigma_1 \cup \sigma_2 \\ A^1, & x \in \sigma_1 \\ B, & x \in \sigma_2 \end{cases} \qquad \text{where } B\psi_1 = \psi_2, \qquad B\psi_2 = \psi_1$$

The measurement of $A_t(x)$ for $x \in \sigma_1$ does change the statistics of the measurements in σ_2: $B\psi = \psi$, while ψ_1 and ψ_2 are not the eigenvectors of B. This clearly contradicts special relativity. However, such observables are not allowed by the relativistic quantum theory: one of the postulates of this theory requires the commutation relation $[A_t(x), A_t(y)] = 0$ for $x \neq y$ to hold (see, for example, Streater and Wightman, 1964, p. 100), whenever $A_t(x)$ corresponds to a directly measurable physical quantity, in our case, $[A^1, B]\psi_1 = (A^1B - BA^1)\psi_1 = 2\psi_2 \neq 0$. Thus no contradiction arises. It is interesting to remark that EPR can be used as a motivation for these commutation relations.

(4) Our final remarks concern quantum logic. In particular, we recall that it is responsible for a motivation of the seemingly ad hoc Hilbert space formalism. A decisive difference between quantum and classical mechanics is due to the existence of noncommuting quantum observables. Accordingly, there exist pairs of propositions of the form $v(A) \in B$ that cannot be verified simultaneously. That is why the internal logic of a quantum system Q, represented by the lattice of all closed subspaces of the Hilbert space X associated with Q, can be best described as a partial Boolean algebra (PBA): one notices that any subset of pairwise "commuting" propositions forms a Boolean algebra (compare Kochen and Specher, 1968).

Definition. A quadruple $(S, \overset{\circ}{\upharpoonleft}, \cap, ')$ is called a *partial Boolean* algebra, if the set S, the binary predicate $\overset{\circ}{\upharpoonleft}$, the binary operation $\cap$, and the unary operation $'$ satisfy the following conditions: (1) any subset $T \subset S$ such that $a \overset{\circ}{\upharpoonleft} b$ is true for any pair $a, b \in T$ forms a Boolean algebra with respect to $\cap, '$; (2) the predicate $\overset{\circ}{\upharpoonleft}$ is reflexive, that is, $a \overset{\circ}{\upharpoonleft} a$ holds, and symmetric; (3) there exist two elements $0, 1$ such that $0 \overset{\circ}{\upharpoonleft} a$, $1 \overset{\circ}{\upharpoonleft} a$ for any a and $a \cap 0 = 0$, $a \cap 1 = a$, $0' = 1$.

The set of all projectors $\mathcal{P}(X)$ in a Hilbert space X forms a PBA, if one sets $P_1 \overset{\circ}{\upharpoonleft} P_2$ whenever $P_1P_2 = P_2P_1$, $P_1 \cap P_2 = \inf(P_1, P_2)$ (in other words, $P_1 \cap P_2$ is the projector on the subspace $P_1X \cap P_2X$), $P_1' = 1 - P_1$ for any $P_1, P_2 \in \mathcal{P}(X)$.

In any PBA we can define a partial ordering $\leqslant$ by $a \leqslant b$, if $a \overset{\circ}{\upharpoonleft} b$ and $a \cap b = a$. A PBA is called a *lattice* if $a \cap b = \text{sub}\{c \mid c \leqslant a, c \leqslant b\}$ exists for any pair a, b of its elements. A lattice is said to be *modular* if $a \cup (b \cap c) = (a \cup b) \cap c$ whenever $a \leqslant c$; it is said to be *orthomodular*, if $a \cup (b \cap a') = b$ for $a \leqslant b$; a minimal element $p \neq 0$ is called an *atom*, a lattice is *atomic*, if for any b there exists an atom $p \leqslant b$. The lattice of *all* subspaces of a vector space is a modular atomic lattice; the lattice of all *closed* subspaces of a Hilbert space is not modular, but it is orthomodular and atomic; the lattice of all *subsets* of a set is a Boolean algebra. A lattice is called *complete*, if every its subset has the least upper bound. It turns out that (Piron, 1964) any *complete orthomodular atomic lattice* is isomorphic to a direct sum of lattices each of which can be realized as a lattice of closed subspaces of a vector space over a field with involution (with possible exceptions in dimension two). This theorem partly explains how the Hilbert space formalism arises and suggests its possible generalizations (Finkelstein et al., 1962). Thus the internal logic $J(Q)$ of a quantum system Q can be described axiomatically as follows: let $\mathfrak{A}$ be the set of observables of Q, then (1) for any $A \in \mathfrak{A}$, $B \in \mathfrak{B}(R)$ the expression $v(A) \in B$ is called an *elementary proposition*; (2) every elementary proposition is a *preformula*; if a, b are preformulas, then $a', a \cup b, a \cap b$ are preformulas; (3) for any preformulas a, b the expression $a \leqslant b$ is a *formula*. Moreover, we introduce the following axioms: (a) $a'' = a$; (b) $(a \cup b)' = a' \cap b'$; (c) $a \cup (b \cap a') = b$ whenever $a \leqslant b$ (orhomodularity). *Semantics* of this system in terms of experimentally verifiable assertions can be built up as follows. Let us consider a classical predicate calculus K with two sets of variables: $a, b, c, \ldots$ for propositions, and $\rho, \rho_1, \rho_2, \ldots$ for measurements and two binary predicates, $T(a, \rho)$ and $H(a, \rho)$. These predicates are to be directly verifiable; $T(a, \rho)$ asserts: "the proposition a has been found to be true by the measurement ρ", $H(a, \rho)$ asserts: "the measurement ρ is performed to verify the validity of a." The following axioms are assumed to hold:

(0) $$T(a, \rho) \supset H(a, \rho)$$

(1) $$H(a, \rho) \supset (T(a, \rho) \vee T(a', \rho))$$

(2) $$\forall a\, \exists \rho H(a, \rho)$$

(3) $$H(a, \rho) \equiv H(a', \rho)$$

To assign a truth value to a formula $a \leqslant b$ of $J(Q)$ we interpret it as a formula $\forall \rho (T(a, \rho) \supset T(b, \rho))$ in K; if a is a preformula of the form $v(A) \in E$, it is assumed to be verified experimentally; $T(a', \rho)$ is interpreted as $H(a, \rho) \& \neg T(a, \rho)$; $a \cup b \leqslant c$ means $a \leqslant c \& b \leqslant c$; $c \leqslant a \cup b$ means $\forall d((a \leqslant d \& b \leqslant d) \supset c \leqslant a \cap b)$; $c \leqslant a \cap b$ means $c \leqslant a \& c \leqslant b$; $a \cap b \leqslant c$ means $\forall d((d \leqslant a \& d \leqslant b) \supset d \leqslant c)$. It follows that $T(a'', \rho)$ can be interpreted as $T(a, \rho)$. Thus every formula $\mathfrak{A}$ of $J(Q)$ can be interpreted as a formula $f(\mathfrak{A})$ in K in such a way that $f(\mathfrak{A})$ contains predicates $T(a, \rho)$ and $H(a, \rho)$ with a of the form $v(A) \in E$ only; we conclude therefore that $f(\mathfrak{A})$ can be verified experimentally. It follows from the above discussion that the quantum mechanical description differs from the classical one by its logicomathematical structure or, alternatively, by the algebraic structure of the set of quantum observables. Indeterminism of quantum mechanics can be considered as a necessary consequence of its underlying mathematical structure. It is tempting to consider quantum mechanics as an incomplete theory and to "discover" the phase space of "hidden variables" related to quantum mechanics in the same way as the phase space of statistical physics is related to macrothermodynamics. The main obstacle is apparent: quantum logic differs from the classical one. However, it is, of course, possible to introduce such variables for each Boolean subalgebra. In fact, the semantical interpretation of quantum logic we have just discussed provides us with an example of such a structure. Let $\mathfrak{M}$ be the set of all measurements and $S_a = \{\rho | T(a, \rho)$ holds$\}$ is the subset of this "phase space" corresponding to the proposition a. It follows from our axioms that $S_{a\cap b} = S_a \cap S_b$, $S_{a\cup b} \leq S_a \cup S_b$; however, in general, $S_{a\cup b} \neq S_a \cup S_b$; moreover, $S_a \cup S_{a'} \neq \mathfrak{M}$, since $S_a \cup S_{a'} = \{\rho | H(a, \rho)$ holds$\}$ and there are measurements which do not check a. Here the difference between quantum and classical theory manifests itself in the impossibility of simultaneous observation for a pair of incompatible propositions. On the other hand, for every Boolean subalgebra of propositions we can set $\mathfrak{M}_B = \{\rho | H(a, \rho)$ for $a \in B\}$; $S_a = \{\rho | T(a, \rho)$ holds, $\rho \in \mathfrak{M}_B\}$ for any $a \in B$. Since for any ρ in $\mathfrak{M}_B$ the formula $T(a, \rho) \vee T(a', \rho)$ holds, it follows that $S_{a\cup b} = S_a \cup S_b$. Thus we have constructed a "phase space" $\mathfrak{M}_B$ for every Boolean subalgebra $B \subset \mathfrak{M}$: one may say that for every $\rho \in \mathfrak{M}_B$ the outcome of the measurements is certain. This construction should be compared with the following general result (Gudder, 1970). Let $\mathfrak{L}$ be a lattice with an orthocomplement,

$$Q = \left\{ m | m: \mathfrak{L} \to [0, 1], m\left(\bigcup_{i=1}^{\infty} a_i \right) = \sum_{i=1}^{\infty} m(a_i) \text{ for pairwise disjoint } a_i \right\}$$

be the set of additive functionals on $\mathfrak{L}$. One can associate with $\mathfrak{L}$ a "phase space" Ω in such a way that for every Boolean subalgebra $B \subset \mathfrak{L}$ there exists a function G_B: $\mathfrak{M} \times \Omega \times B \to \{0, 1\}$ ($\{0, 1\}$ is the set containing two elements 0, 1 only), and $G_B(m, \omega, a_1) \leq G_B(m, \omega, a_2)$ whenever $a_1 \leq a_2$. We say that a holds in the "phase space" point (m, ω) whenever $G_B(m, \omega, a) = 1$. One can prove, however, that for a non-Boolean $\mathfrak{L}$ the function G_B does depend on B (see Kochen and Specker, 1968, for precise theorems of this kind), therefore the proposition a may be valid if checked in the context of B and false in the context B' [i.e., $G_{B'}(m, \omega, a) = 1$, while $G_B(m, \omega, a) = 0$]. In our construction a is checked independently of the context, but the phase space $\mathfrak{M}_B$ is context dependent. As numerous impossibility theorems (e.g., Kochen and Specker, 1968) show, the deterministic paradise of classical physics is lost, and one cannot mimic the statistical physics phase space approach to restore it.

(5) To close this brief outline of a few problems and ideas in foundations of quantum mechanics we have to note that the measurement of

observables as described above does not require any "reduction of the wave packet" or similar conceptions. Quantum theory cannot make any a priori predictions about behavior of a microsystem after its interaction with a macroscopic measuring device. Such an interaction is to be analyzed individually according to the laws of quantum mechanics. A rather complete bibliography on the foundations of quantum mechanics is to be found in Suppers (1976). This paper is based on our studies of quantum logic (Moroz, 1971, 1974, 1977, 1979; Moroz and Kreinovich, 1975). We are influenced and guided by the publications listed in the References.

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REFERENCES

Band, W., and Park, J. L. (1979). *American Journal of Physics*, **47**, 188–191.

Benioff, P. A. (1977). *Journal of Mathematical Physics*, **18**, 2289–2295.

Birkhoff, G., and von Neumann, J. (1936). *Annals of Mathematics*, **37**, 823–843.

Einstein, A., Podolsky, B., and Rosen, N. (1935). *Physical Review*, **47**, 777–780.

Finkelstein, D., Jauch, J. M., Schiminovich, S., and Speiser, D. (1962). *Journal of Mathematical Physics*, **3**, 431–436.

Gudder, S. (1970). *Journal of Mathematical Physics*, **11**, 431–436.

Heisenberg, W. (1930). *The Physical Principles of the Quantum Theory*. Dover, New York.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Kemble, E. C. (1937). *Fundamental Principles of Quantum mechanics*. McGraw Hill, New York.

Kochen, S., and Specker, E. (1968). *Journal of Mathematics and Mechanics*, **17**, 59–87.

Kreinovich, V. Ja. (1977). *Theoretical and Mathematical Physics*, **28**, 628–633.

Moroz, B. Z. (1971). *Soviet Mathematics Doklady*, **12**, 934–937.

Moroz, B. Z. (1974). *Seminars in Mathematics, V. A. Steklov Mathematical Institute*, **39**, 189–190.

Moroz, B. Z. (1977). Tel Aviv University Preprint, TAUP 597-77.

Moroz, B. Z. (1979). Ben Gurion University of Negev Preprint.

Moroz, B. Z., and Kreinovich, V. Ja. (1975). Proceedings of the 5th Int. Congress in logic, philosophy and methodology of Science, London, Ontario, 1975.

Piron, C. (1964). "Axiomatique Quantique," Thèse, Univ. de Lausanne.

Reichenbach, H. (1948). *Philosophical Foundations of Quantum Mechanics*. Univ. of California Press, Berkeley.

Schrödinger, E. (1935). *Naturwissenschaften*, **23**, 807–812.

Streater, R. P., and Wightmann, A. S. (1964). *PCT, Spin, Statistics and All That*. Benjamin, New York.

Suppers, P. (1976). *Logic and Probability in Quantum Mechanics*, D. Reidel, Dordrecht.

Varadarajan, V. C. (1968). *Geometry of quantum mechanics*, Vols. I and II. Van Nostrand, New York.

von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton Univ. Press, Princeton, New Jersey.